Relativistic Coulomb Green's function in d-dimensions.

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(Dated: January 31, 2011)

Abstract

Using the operator method, the Green's functions of the Dirac and Klein-Gordon equations in the Coulomb potential $-Z\alpha/r$ are derived for the arbitrary space dimensionality d. Nonrelativistic and quasiclassical asymptotics of these Green's functions are considered in detail.

I. INTRODUCTION

When calculating the amplitudes and probabilities of QED processes in the field of heavy atoms, one should be aware that the parameter $Z\alpha$ (Z is the atomic charge number and α is the fine-structure constant) is not small in this case. The effect of higher orders in $Z\alpha$ can change the Born result by several times. Therefore, it is often required to calculate the probabilities of QED processes in such a strong field exactly in $Z\alpha$. The most convenient way to perform this calculation is the use of the exact Green's functions of the Dirac equation (or Klein-Gordon equation) for a charged particle in the field (Furry representation). Therefore, derivation of the Green's functions for specific field configurations is very important for applications. For the case of the Coulomb potential, a convenient integral representation of the Green's function $G(\boldsymbol{r}, \boldsymbol{r}'|\varepsilon)$ was derived in Ref. [1] using the $O(2,1)$ algebra. The representation obtained is valid in the whole complex plane of the energy ε and does not contain contour integrals. Another integral representation for the Green's function in a Coulomb field was derived in Ref.[2] by using the explicit form of the expansion of $G(\boldsymbol{r}, \boldsymbol{r}'|\varepsilon)$ with respect to the eigenfunctions of the corresponding wave equation. The representation of the Green's function obtained in Ref.[2] contains a contour integral, which complicates its use in applications.

In the calculation of the loop diagrams, it is often required to regularize the divergent integrals. One of the most convenient methods of the regularization is the dimensional regularization. In order to use the dimensional regularization within the approach based on the Furry representation, it is necessary to derive the exact Green's function in the Coulomb field in arbitrary, not necessarily integer, space dimensionality d (the space-time dimensionality is $d+1$). In the present paper, we solve this problem by generalizing the Green's function, obtained in Ref. [1] for $d=3$, to arbitrary d. Our derivation closely follows the path of derivation in Ref. [1]. In contrast to the conventional approach, the operator method used in Ref. [1] and in the present paper does not require the knowledge of the explicit form of the wave functions, which is difficult to define for non-integer d. In order to fix unambiguously the explicit form of the Green's function for arbitrary d, we use only the commutative and anticommutative relations for the operators and γ-matrices .

II. CALCULATION OF THE GREEN'S FUNCTION

Following Ref. [1], we represent the Green's function in the Coulomb potential $U(r) = -Z\alpha/r$ (the system of units $\hbar = c = 1$ is used),

$$G(\boldsymbol{r},\boldsymbol{r}'|\varepsilon) = \frac{1}{\hat{\mathcal{P}} - m + i0}\delta(\boldsymbol{r}-\boldsymbol{r}')\,, \quad \hat{\mathcal{P}} = \gamma^0(\varepsilon + Z\alpha/r) - \boldsymbol{\gamma}\boldsymbol{p}\,, \tag{1}$$

as follows

$$\begin{aligned}
G(\boldsymbol{r},\boldsymbol{r}'|\varepsilon) &= (\hat{\mathcal{P}}+m)D(\boldsymbol{r},\boldsymbol{r}'|\varepsilon)\\
D(\boldsymbol{r},\boldsymbol{r}'|\varepsilon) &= -i\int_0^\infty ds\exp\left\{2iZ\alpha\varepsilon s - is\left[rp_r^2 + \kappa^2 r + \frac{K}{r}\right]\right\}\frac{\delta(r-r')}{r^{d-2}}\delta(\boldsymbol{n}-\boldsymbol{n}')\,,\\
&\kappa = \sqrt{m^2-\varepsilon^2}\,,\quad p_r = -\frac{i}{r^{(d-1)/2}}\frac{\partial}{\partial r}r^{(d-1)/2}\,,\quad \boldsymbol{n} = \boldsymbol{r}/r\,,\quad \boldsymbol{n}' = \boldsymbol{r}'/r'\,,\\
&K = \boldsymbol{l}^2 - iZ\alpha\boldsymbol{\alpha}\boldsymbol{n} - (Z\alpha)^2 + \frac{1}{4}(d-1)(d-3)\,,\quad \boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}\,.
\end{aligned} \tag{2}$$

Here $-\boldsymbol{l}^2$ is the angular part of Laplacian determined by

$$\Delta = \frac{1}{r^{d-1}}\partial_r r^{d-1}\partial_r - \frac{1}{r^2}\boldsymbol{l}^2\,. \tag{3}$$

and γ-matrices obey the usual relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = g^{\mu\nu}$.

Then we represent the angular part of the δ-function as

$$\delta(\boldsymbol{n}-\boldsymbol{n}') = \sum_\lambda P_\lambda(\boldsymbol{n},\boldsymbol{n}')\,, \tag{4}$$

where the projection operators $P_\lambda(\boldsymbol{n},\boldsymbol{n}')$ satisfy the relations

$$\begin{aligned}
&K\,P_\lambda(\boldsymbol{n},\boldsymbol{n}') = \lambda(\lambda+1)P_\lambda(\boldsymbol{n},\boldsymbol{n}')\,,\\
&\int d\boldsymbol{n}' P_\lambda(\boldsymbol{n},\boldsymbol{n}')P_{\lambda'}(\boldsymbol{n}',\boldsymbol{n}'') = \delta_{\lambda\lambda'}P_\lambda(\boldsymbol{n},\boldsymbol{n}'')\,.
\end{aligned} \tag{5}$$

Since the operator K contains only one matrix operator $\boldsymbol{\alpha}\boldsymbol{n}$, the matrix structure of the projection operator $P_\lambda(\boldsymbol{n},\boldsymbol{n}')$ is given by the linear combination of the unit matrix I and matrices $\boldsymbol{\alpha}\boldsymbol{n}$, $\boldsymbol{\alpha}\boldsymbol{n}'$, and $(\boldsymbol{\alpha}\boldsymbol{n})(\boldsymbol{\alpha}\boldsymbol{n}')$. All other matrices, such as $(\boldsymbol{\alpha}\boldsymbol{n})(\boldsymbol{\alpha}\boldsymbol{n}')(\boldsymbol{\alpha}\boldsymbol{n})$, can be reduced to the four above mentioned matrices using the anticommutative relations. Taking this property into account, we search the projection operators $P_\lambda(\boldsymbol{n},\boldsymbol{n}')$ in the form

$$\begin{aligned}
P_\lambda(\boldsymbol{n},\boldsymbol{n}') &= a_1\Lambda_+(\boldsymbol{n})\Lambda_+(\boldsymbol{n}') + a_2\Lambda_+(\boldsymbol{n})\Lambda_-(\boldsymbol{n}')\\
&\quad + a_3\Lambda_-(\boldsymbol{n})\Lambda_+(\boldsymbol{n}') + a_4\Lambda_-(\boldsymbol{n})\Lambda_-(\boldsymbol{n}')\,,\\
&\quad \Lambda_\pm(\boldsymbol{n}) = \frac{1}{2}(1\pm\boldsymbol{\alpha}\boldsymbol{n})\,,
\end{aligned} \tag{6}$$

where a_i are some functions of $x = \boldsymbol{n}\boldsymbol{n}'$. From Eqs. (5), we obtain

$$
\begin{aligned}
& a_1 = \beta(\lambda + iZ\alpha)B_n(x)\,, \quad a_2 = a_3 = \beta\,(n+\nu+1/2)\,A_n(x)\,, \quad a_4 = \beta(\lambda - iZ\alpha)B_n(x)\,, \\
& \lambda = \pm\gamma\,, \quad \gamma = \sqrt{(n+\nu+1/2)^2 - (Z\alpha)^2}\,, \quad \beta = \frac{\Gamma(\nu+1)}{2\lambda\pi^{\nu+1}}\,, \\
& A_n(x) = \frac{1}{2\nu}\frac{\partial}{\partial x}[C^\nu_{n+1}(x) + C^\nu_n(x)]\,, \quad B_n(x) = \frac{1}{2\nu}\frac{\partial}{\partial x}[C^\nu_{n+1}(x) - C^\nu_n(x)]\,, \\
& \nu = \frac{d}{2} - 1\,,
\end{aligned}
\tag{7}
$$

where $C^\nu_n(x)$ is the Gegenbauer polynomial, and $n = 0, 1, 2, \ldots$ is integer number. This integer number appears from the requirement that the functions a_i have no singularities at $x = 1$. The result (7) for a_i was obtained with the help of the identity

$$
\begin{aligned}
& \int (1 + \boldsymbol{n}\boldsymbol{n}' + \boldsymbol{n}\boldsymbol{n}'' + \boldsymbol{n}'\boldsymbol{n}'')B_n(\boldsymbol{n}\boldsymbol{n}')B_n(\boldsymbol{n}'\boldsymbol{n}'')\,d\boldsymbol{n}' = \Omega_d(1 + \boldsymbol{n}\boldsymbol{n}'')B_n(\boldsymbol{n}\boldsymbol{n}'')\,, \\
& \Omega_d = \int d\boldsymbol{n} = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \frac{2\pi^{\nu+1}}{\Gamma(\nu+1)}\,,
\end{aligned}
\tag{8}
$$

Finally we obtain for projection operator

$$
\begin{aligned}
P_\lambda(\boldsymbol{n}, \boldsymbol{n}') \;=\; & \frac{\beta}{2}\Big\{ \Big[\lambda[1 + (\boldsymbol{\alpha}\boldsymbol{n})(\boldsymbol{\alpha}\boldsymbol{n}')] + iZ\alpha(\boldsymbol{\alpha}\boldsymbol{n} + \boldsymbol{\alpha}\boldsymbol{n}')\Big] B_n(x) \\
& + (n+\nu+1/2)\,[1 - (\boldsymbol{\alpha}\boldsymbol{n})(\boldsymbol{\alpha}\boldsymbol{n}')]A_n(x) \Big\}\,.
\end{aligned}
\tag{9}
$$

For $d = 3$, this projection operator coincides with that found in Ref.[3].

Note that the functions $A_n(x)$ and $B_n(x)$ have non-singular limit at $\nu \to 0$ (or $d \to 2$) ,

$$
\lim_{\nu\to 0} A_n(x) = \frac{\sin((n+1)\phi) + \sin(n\phi)}{\sin\phi}\,, \quad \lim_{\nu\to 0} B_n(x) = \frac{\sin((n+1)\phi) - \sin(n\phi)}{\sin\phi}\,,
$$

with $\phi = \arccos x$.

In order to complete the calculation of the function $D(\boldsymbol{r}, \boldsymbol{r}'|\varepsilon)$, Eq. (2), it is necessary to find the result of the action of the operator $\exp\{-is\,[rp_r^2 + \kappa^2 r + \lambda(\lambda+1)/r]\}$ on the function $\delta(r - r')/r^{2\nu}$. This can be done exactly in the same way as in Ref.[1]. The method of Ref.[1] is based on the commutator relations of the operators $T_1 = \frac{1}{2}[rp_r^2 + \lambda(\lambda+1)/r]$, $T_2 = rp_r$, and $T_3 = r$ which coincide with those of the $O(2,1)$ algebra generators (some other examples of applying the $O(2,1)$ algebra in a Coulomb field can be found in Refs.[4, 5]). The only difference between the case of arbitrary d and $d = 3$ is the value of the parameter δ in the equation $T_1 r^\delta = 0$. For arbitrary d, we have

$$
\delta = \lambda + \frac{3-d}{2} \quad \text{at}\ \lambda > 0\,, \quad \delta = |\lambda| + \frac{1-d}{2} \quad \text{at}\ \lambda < 0\,. \tag{10}
$$

The final result for the function $D(\boldsymbol{r},\boldsymbol{r}'|\varepsilon)$ in Eq.(2) reads

$$
\begin{aligned}
D(\boldsymbol{r},\boldsymbol{r}'|\varepsilon) \;&=\; -\frac{i\Gamma\left(\nu+1\right)}{2\pi^{\nu+1}(rr')^{\nu+1/2}}\sum_{n=0}^{\infty}\int_0^{\infty} ds\exp[2iZ\alpha\varepsilon s+i\kappa(r+r')\cot(\kappa s)-i\pi\gamma] \\
&\times\left\{\frac{y}{2}J'_{2\gamma}(y)[1+(\boldsymbol{\alpha n})(\boldsymbol{\alpha n}')]B_n(x)+iZ\alpha J_{2\gamma}(y)(\boldsymbol{\alpha n}+\boldsymbol{\alpha n}')B_n(x)\right. \\
&\left.+\,(n+\nu+1/2)\,J_{2\gamma}(y)[1-(\boldsymbol{\alpha n})(\boldsymbol{\alpha n}')]A_n(x)\right\}, \quad y=\frac{2\kappa\sqrt{rr'}}{\sin(\kappa s)}\,, \qquad (11)
\end{aligned}
$$

where $J_{2\gamma}(y)$ is the Bessel function, $A_n(x)$, $B_n(x)$, ν and γ are defined in Eq.(7). The corresponding result for the Coulomb Green's function of the Dirac equation in d space dimension has the form

$$
\begin{aligned}
&G(\boldsymbol{r},\boldsymbol{r}'|\varepsilon) = -\frac{i\Gamma\left(\nu+1\right)}{2\pi^{\nu+1}(rr')^{\nu+1/2}}\sum_{n=0}^{\infty}\int\limits_0^{\infty} ds\exp[2iZ\alpha\varepsilon s+i\kappa(r+r')\cot(\kappa s)-i\pi\gamma]\,\mathcal{T} \\
&\mathcal{T}=[1+(\boldsymbol{\alpha n})(\boldsymbol{\alpha n}')]\left[\frac{y}{2}J'_{2\gamma}(y)(\gamma^0\varepsilon+m)-iZ\alpha J_{2\gamma}(y)\gamma^0\kappa\cot(\kappa s)\right]B_n(x) \\
&+\Big[[1-(\boldsymbol{\alpha n})(\boldsymbol{\alpha n}')](\gamma^0\varepsilon+m)-\kappa\cot(\kappa s)(\boldsymbol{\gamma n}-\boldsymbol{\gamma n}')\Big]J_{2\gamma}(y)\,(n+\nu+1/2)\,A_n(x) \\
&+\left[\frac{i\kappa^2(r-r')}{2\sin^2(\kappa s)}+imZ\alpha\gamma^0\right](\boldsymbol{\gamma n}+\boldsymbol{\gamma n}')J_{2\gamma}(y)B_n(x)\,. \qquad (12)
\end{aligned}
$$

For $d=3$ this result coincides with the corresponding result of Ref.[1]. The function $G(\boldsymbol{r},\boldsymbol{r}'|\,\varepsilon)$ has, in the complex plane ε, cuts along the real axis from $-\infty$ to $-m$ and from m to ∞, which correspond to the continuous spectrum, and has also simple poles in the interval $(0,m)$ for an attractive field and in the interval $(-m,0)$ for a repulsive field. The integral representation (12) is valid for any ε which belongs to the domain $\mathrm{Re}\,\varepsilon<0$, $\mathrm{Im}\,\varepsilon<0$ or $\mathrm{Re}\,\varepsilon>0$, $\mathrm{Im}\,\varepsilon>0$. If $\mathrm{Re}\,\varepsilon<0$, $\mathrm{Im}\,\varepsilon>0$ or $\mathrm{Re}\,\varepsilon>0$, $\mathrm{Im}\,\varepsilon<0$, then it is necessary to perform the integration over s in Eq.(12) from zero to $-\infty$.

For real ε in the interval $-m<\varepsilon<m$ we obtain (cf. Ref.[1])

$$
\begin{aligned}
&G(\boldsymbol{r},\boldsymbol{r}'|\varepsilon) = \frac{\Gamma\left(\nu+1\right)}{4\kappa\sin[\pi(Z\alpha\varepsilon/\kappa-\gamma)]\pi^{\nu+1}(rr')^{\nu+1/2}}\sum_{n=0}^{\infty}\int\limits_{-\pi/2}^{\pi/2} ds \\
&\times\exp\left[-2iZ\alpha\varepsilon s/\kappa+i\kappa(r+r')\tan s\right]\,\mathcal{T} \\
&\mathcal{T}=[1+(\boldsymbol{\alpha n})(\boldsymbol{\alpha n}')]\left[\frac{v}{2}J'_{2\gamma}(v)(\gamma^0\varepsilon+m)-iZ\alpha J_{2\gamma}(v)\gamma^0\kappa\tan s\right]B_n(x) \\
&+\Big[[1-(\boldsymbol{\alpha n})(\boldsymbol{\alpha n}')](\gamma^0\varepsilon+m)-\kappa\tan s(\boldsymbol{\gamma n}-\boldsymbol{\gamma n}')\Big]J_{2\gamma}(v)\,(n+\nu+1/2)\,A_n(x) \\
&+\left[\frac{i\kappa^2(r-r')}{2\cos^2 s}+imZ\alpha\gamma^0\right](\boldsymbol{\gamma n}+\boldsymbol{\gamma n}')J_{2\gamma}(v)B_n(x)\,,\quad v=\frac{2\kappa\sqrt{rr'}}{\cos s}\,. \qquad (13)
\end{aligned}
$$

The denominator in Eq. (13) is zero at points $Z\alpha\varepsilon/\kappa - \gamma = k$ for any integer k. However, the integral over s also vanishes for negative k at these points (see Ref.[1]) so that the expression(13) has poles only for $k = 0,\, 1,\, 2\ldots$. Taking into account that γ is positive, we find that the simple poles corresponding to the discrete spectrum are at the points

$$\varepsilon = \frac{m\,\mathrm{sign}Z}{\sqrt{1+\left(\frac{Z\alpha}{k+\gamma}\right)^2}}\,. \tag{14}$$

The maximal value of Z when all the results obtained are applicable is determined by the relation $(Z\alpha)_{max} = (d-1)/2$ (see the definition of γ in Eq.(7)).

For completeness, we also present the final result for the Coulomb Green's function of the Klein-Gordon equation,

$$\begin{aligned} G_0(\boldsymbol{r},\boldsymbol{r}'|\varepsilon) = &-\frac{\Gamma\,(\nu+1)}{2\pi^{\nu+1}(rr')^{\nu}}\sum_{n=0}^{\infty}\frac{n+\nu}{\nu}\,C_n^{\nu}(x) \\ \times &\int_0^{\infty}\frac{\kappa\,ds}{\sin(\kappa s)}\exp[2iZ\alpha\varepsilon s + i\kappa(r+r')\cot(\kappa s) - i\pi\mu]J_{2\mu}(y)\,, \\ \mu = &\sqrt{(n+\nu)^2-(Z\alpha)^2}\,. \end{aligned} \tag{15}$$

Note that there is no singularity in this formula at $d = 2$ because

$$\lim_{\nu\to 0}\frac{n+\nu}{\nu}\,C_n^{\nu}(x) = \cos(n\phi)\,.$$

III. ASYMPTOTICS

Let us derive the Coulomb Green's function $G_{nr}(\boldsymbol{r},\boldsymbol{r}'|E)$ of the Schrödinger equation for d space dimensions. In order to do this we calculate the nonrelativistic asymptotics of the Coulomb Green's function of the Klein-Gordon equation valid at $|E| \ll m$ and $Z\alpha \ll 1$, where $E = \varepsilon - m$. Neglecting $(Z\alpha)^2$ in μ, using the formula of summation (cf. [2]),

$$\begin{aligned} S_0 &= \sum_{n=0}^{\infty}(-1)^n\frac{\nu+n}{\nu}C_n^{\nu}(x)J_{2(n+\nu)}(y) \\ &= \frac{\sqrt{\pi}\,y^{2\nu}J_{\nu-1/2}(w)}{2^{3\nu+1/2}\Gamma(\nu+1)w^{\nu-1/2}}\,, \quad w = y\sqrt{\frac{1+x}{2}} \end{aligned} \tag{16}$$

and multiplying by $2m$, we obtain

$$G_{nr}(\boldsymbol{r},\boldsymbol{r}'|E) = -\frac{m}{(2\pi)^{\nu+1/2}}\int_0^{\infty}ds\left(\frac{\kappa}{\sin(\kappa s)}\right)^{2\nu+1}$$

$$
\begin{aligned}
&\times \exp[2iZ\alpha ms + i\kappa(r+r')\cot(\kappa s) - i\pi\nu]\frac{J_{\nu-1/2}(w)}{w^{\nu-1/2}}\,, \\
&\kappa = \sqrt{-2mE}\,. \qquad (17)
\end{aligned}
$$

This formula is in agreement with the corresponding result of Ref.[6].

At high energies and small scattering angles of the particles, the characteristic angular momenta are large and the quasiclassical approximation is applicable. The quasiclassical Green's function of the Dirac equation in a Coulomb potential for $d = 3$ was first derived in Refs. [7, 8]. Another representation of this function was obtained in Refs. [9, 10]. The quasiclassical Green's function for arbitrary spherically symmetric localized potential was found in Refs. [11, 12]. In Ref.[13], the quasiclassical Green's function for arbitrary localized potential was found with the next-to-leading quasiclassical correction taken into account. In Ref.[13], a spherical symmetry of the potential was not required. Let us consider the quasiclassical Green's function of the Dirac equation in a Coulomb potential for arbitrary space dimension d. In this case $\varepsilon \gg m$ and $1 + x \ll 1$ so that the main contribution to the sum over n in Eq.(12) is given by $n \gg 1$. Thus we can neglect the term $(Z\alpha)^2$ in γ, Eq.(7), and perform summation over n analytically. We need to calculate two sums,

$$
\begin{aligned}
S_A &= \sum_{n=0}^{\infty}(-1)^n(\nu+n+1/2)\,A_n(x)J_{2(n+\nu+1/2)}(y)\,, \\
S_B &= \sum_{n=0}^{\infty}(-1)^n B_n(x)J_{2(n+\nu+1/2)}(y)\,, \qquad (18)
\end{aligned}
$$

where the functions $A_n(x)$ and $B_n(x)$ are defined in Eq.(7). Using the recurrent relations for the Bessel functions and the Gegenbauer polynomials, it is easy to show that

$$
S_A = (1+x)\frac{\partial}{\partial x}S_B + (\nu+1/2)S_B\,, \quad S_B = -\frac{2}{y}\frac{\partial}{\partial x}S_0\,, \qquad (19)
$$

so that

$$
S_A = = \frac{\sqrt{\pi}\,y^{2\nu+1}J_{\nu-1/2}(w)}{2^{3\nu+5/2}\Gamma(\nu+1)w^{\nu-1/2}}\,, \quad S_B = \frac{\sqrt{\pi}\,y^{2\nu+1}J_{\nu+1/2}(w)}{2^{3\nu+3/2}\Gamma(\nu+1)w^{\nu+1/2}}\,. \qquad (20)
$$

Substituting these results in Eq.(12), we arrive at the final expression to the quasiclassical Green's function

$$
G_{qc}(\boldsymbol{r},\boldsymbol{r}'|\varepsilon) = -\frac{1}{2^{\nu+3/2}\pi^{\nu+1/2}}\int\limits_0^{\infty}\frac{ds}{u^{\nu-1/2}}\left(\frac{p}{\sinh(ps)}\right)^{2\nu+1}
$$

$$\times \exp[2iZ\alpha\varepsilon s + ip(r+r')\coth(ps) - i\pi\nu]\,\mathcal{M}$$

$$\mathcal{M} = J_{\nu-1/2}(u)\left[\gamma^0\varepsilon + m - \frac{p}{2}\coth(ps)(\boldsymbol{\gamma n} - \boldsymbol{\gamma n}')\right] + i\frac{J_{\nu+1/2}(u)}{u}$$

$$\times\left\{\left[\frac{p^2(r-r')}{2\sinh^2(ps)} + mZ\alpha\gamma^0\right](\boldsymbol{\gamma n} + \boldsymbol{\gamma n}') - Z\alpha\gamma^0 p\coth(ps)[1 + (\boldsymbol{\alpha n})(\boldsymbol{\alpha n}')]\right\},$$

$$u = \frac{p\sqrt{2rr'(1+x)}}{\sinh(ps)}, \tag{21}$$

where $p = \sqrt{\varepsilon^2 - m^2} = i\kappa$. For $d = 3$ the result (21) is in agreement with that obtained in Refs.[7, 8]. The term $(Z\alpha)^2$ in γ, Eq.(7), can be also neglected in the nonrelativistic approximation when $Z\alpha \ll 1$, $p \ll m$, and $Z\alpha m/p$ is fixed. In this case we immediately obtain from Eq.(21) that the nonrelativistic approximation for the Green's function of the Dirac equation reads

$$G(\boldsymbol{r}, \boldsymbol{r}'|m+E) = \frac{\gamma^0 + 1}{2} G_{nr}(\boldsymbol{r}, \boldsymbol{r}'|E)\,, \tag{22}$$

where $G_{nr}(\boldsymbol{r}, \boldsymbol{r}'|\varepsilon)$ is defined in Eq.(17).

To summarize, we have calculated in d space dimensions the Green's functions of the Dirac, Eq.(12), and Klein-Gordon, Eq.(15), equations in the Coulomb field. Nonrelativistic and quasiclassical limiting cases of these Green's functions are considered in detail. The results obtained can be applied for calculation of various QED amplitudes in the strong Coulomb field with the use of dimensional regularization.

This work was supported in part by the RFBR Grant No 09-02-00024 and the Grant 14.740.11.0082 of Federal Program "Personnel of Innovational Russia". I.S.T. was also supported by the "Dynasty" foundation.

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